

A Note on Comparison of Error Correction Codes

Dejan V. Djonin

Abstract

Use of an error correction code in a given transmission channel can be regarded as the statistical experiment. Therefore, powerful results from the theory of comparison of experiments can be applied to compare the performances of different error correction codes. We present results on the comparison of block error correction codes using the representation of error correction code as a linear experiment. In this case the code comparison is based on the Loewner matrix ordering of respective code matrices. Next, we demonstrate the bit-error rate code performance comparison based on the representation of the codes as dichotomies, in which case the comparison is based on the matrix majorization ordering of their respective equivalent code matrices.

Index Terms

Deficiency distance, error-correction code, code design, probability of error, Bayes risk, zonotope, matrix majorization.

I. INTRODUCTION

In this section we will review the basic concepts of statistical experiments that will later be used to establish ordering relation between error correction codes employed in a given communication channel. Following [1], a *statistical experiment* is defined as a pair $\mathcal{E} = (\mathcal{X}, P_\theta; \theta \in \Theta)$ where \mathcal{X} is a measurable *sample space*, P_θ is a probability measure on \mathcal{X} for each $\theta \in \Theta$ and Θ is a *parameter set*. Let the decision rule $y = \sigma(x)$ of experiment \mathcal{E} be defined as a mapping from an element of the sample space $x \in \mathcal{X}$ to the element of the decision space $y \in \mathcal{Y}$. Further, one can define the loss function $\mathcal{L}(\theta, y)$ of choosing the decision $y \in \mathcal{Y}$ when the true state of the parameter is $\theta \in \Theta$. The reader is referred to [1] for a thorough treatment of statistical experiments, loss functions, with numerous illustrative examples. We will next introduce the notation used throughout the paper, followed by the definition and properties of deficiency distance between experiments.

The author is with Dyaptive Systems, Inc., Vancouver, BC and adjunct professor at the Department of Electrical and Computer Engineering, University of British Columbia, 2356 Main Mall, Vancouver, BC, V6T 1Z4, Canada, e-mail: ddjonin@ece.ubc.ca.

Notation: We will use a' to denote the transpose of vector a , A^+ denotes the Moore-Penrose pseudoinverse of the matrix A , $\#S$ denotes the cardinality of set S . $A \wedge B$ and $A \vee B$ denote the element-wise minimum and maximum of two equal dimensional matrices A and B , respectively. Let $A(i, :)$ and $A(:, i)$ denote the i -th row and i -th column of the matrix A , respectively. Range $\text{range}(B)$ of the matrix B is the space spanned by the columns of the matrix B . Let e_i be $[0, 0, \dots, 0, 1, 0, \dots, 0]$ where 1 is at the i -th position of vector e_i . Let $\text{conv}(a, b) = \{\alpha a + (1 - \alpha)b | \alpha \in [0, 1]\}$ for equal dimensional vectors a and b . Let \mathcal{M}_{nm} be the set of row-stochastic $n \times m$ dimensional matrices, i.e. these matrices have positive elements whose rows sum to 1. Let indicator function $I_{\{x\}}$ be equal to 1 if x is true and 0 otherwise. $N(\mu, \Sigma)$ denotes the multivariate normal random variable with mean μ and covariance Σ . $\|f\|$ denotes the total variation of function $f(x)$ given by $\int |f(x)|dx$. \mathbb{E}_u is the expectation with respect to (w.r.t) random variable u .

Le Cam has introduced in [2] the following definition of the *deficiency distance* between two experiments:

Definition 1: (Deficiency Distance) Experiment $\mathcal{E} = (\mathcal{X}, P_\theta; \theta \in \Theta)$ is ϵ -deficient w.r.t experiment $\mathcal{F} = (\mathcal{Y}, Q_\theta; \theta \in \Theta)$ if for each $\Theta_0 \subset \Theta$, and each decision rule ρ of experiment \mathcal{F} , there exists a decision rule σ of experiment \mathcal{E} such that

$$\int P_\theta(x) \mathcal{L}(\theta, \sigma(x)) dx \leq \int Q_\theta(y) \mathcal{L}(\theta, \rho(y)) dy + \epsilon_\theta \max_y \mathcal{L}(\theta, y) \quad (1)$$

when $\theta \in \Theta_0$. In short form, the deficiency distance is denoted as $\epsilon = \delta(\mathcal{E}, \mathcal{F})$.

The deficiency distance can plausibly be interpreted as follows: the deficiency of experiment \mathcal{E} w.r.t the experiment \mathcal{F} is the upper bound on the difference in the risk functions¹ between experiments \mathcal{E} and \mathcal{F} for any choice of the a priori knowledge of the unknown parameter θ and some decision rule on \mathcal{E} . In general, therefore, the deficiency distance is *not symmetric*, i.e. $\delta(\mathcal{E}, \mathcal{F}) \neq \delta(\mathcal{F}, \mathcal{E})$.

Historically, the deficiency distance is a generalization of the older concept of sufficiency ordering of experiments.

Definition 2: (Sufficient Experiments [3], [1]) Experiment $\mathcal{E} = (\mathcal{X}; P_\theta, \theta \in \Theta)$ is sufficient for experiment $\mathcal{F} = (\mathcal{Y}; Q_\theta, \theta \in \Theta)$ iff $\delta(\mathcal{E}, \mathcal{F}) = 0$. The ordering of experiments based on sufficiency is also denoted by $\mathcal{E} \geq_S \mathcal{F}$.

¹Expression $\int P_\theta(x) \mathcal{L}(\theta, \sigma(x)) dx$ is commonly referred as the *risk function* of the experiment $\mathcal{E} = (\mathcal{X}, P_\theta; \theta \in \Theta)$ for a given loss function L and decision rule σ .

Note that ordering \geq_S is a partial ordering on the set of experiments with the same parameter set. This ordering is sometimes also referred to as Blackwell ordering. We next discuss the basic properties of the deficiency distance.

Theorem 1: (Properties of Deficiency Distance) Let $\mathcal{E}, \mathcal{F}, \mathcal{G}$ be experiments. Then

$$(i) \ 0 \leq \delta(\mathcal{E}, \mathcal{F}) \leq 2 - 2(\#\Theta)^{-1}$$

$$(ii) \ \delta(\mathcal{E}, \mathcal{G}) \leq \delta(\mathcal{E}, \mathcal{F}) + \delta(\mathcal{F}, \mathcal{G})$$

$$(iii) \ \delta(\mathcal{E}, \mathcal{E}) = 0$$

Proof: Theorem 6.2.24 in [1]. \square

A special case of a statistical experiment is a linear experiment defined below²:

Definition 3: (Linear Normal Experiment) Linear normal experiment \mathcal{E} is denoted as $\mathcal{E}(A, \Sigma; \beta \in \mathcal{R}^n)$ and is experiment with the sample distribution $N(A'\beta, \Sigma)$ where the parameter is $\beta \in \mathcal{R}^n$ and the parameter set is \mathcal{R}^n . Real valued matrices A and Σ are assumed to be known.

Two linear experiments can be compared as follows.

Theorem 2: (Linear Experiment Comparison) (a) Linear normal experiment $\mathcal{E}(A, \Sigma; \beta \in \mathcal{R}^n)$ is sufficient for $\mathcal{F}(B, \Gamma; \beta \in \mathcal{R}^n)$ (i.e. $\delta(\mathcal{E}, \mathcal{F}) = 0$) if

$$A\Sigma A' - B\Gamma B' \text{ is non-negative definite matrix.} \quad (2)$$

(b) Let $\Sigma = I$ and $\Gamma = I$ then $\delta(\mathcal{E}, \mathcal{F}) < 2$ iff $\text{range}(B) \subseteq \text{range}(A)$. Furthermore, if $\text{range}(B) \subseteq \text{range}(A)$ then

$$\delta(\mathcal{E}, \mathcal{F}) = \|N(0, (B'(AA')^+ B) \vee I) - N(0, I)\|. \quad (3)$$

Proof: Theorem 8.2.13 and 8.5.7 in [1]. \square

General statistical results and inequalities can provide important insights into the operation and efficient design of a communication system. An early result on the influence of the communication channels based on Blackwell ordering of channel transfer matrices has been presented [4]. More recently, monotonicity results on the influence of Rician fading on the capacity of MIMO systems has been demonstrated in [5]. Based on general inequalities of stochastic majorization, performance comparison of various receivers in multipath fading channels and the influence of power delay profile has been shown in [6].

Starting with the general notion of deficiency, we will present in the next section the application of these concepts to the comparison of error correction codes. Namely, in Subsection II-A we present the

²A more general definition of linear experiment may be given as in [1], Section 8.

block error correction code comparison in additive white Gaussian Noise (AWGN) channels based on linear experiments and block-error rates. In Subsection II-B we present the bit-error rate comparison of error-correction codes used in discrete channels based on matrix majorization.

II. ERROR CORRECTION BLOCK CODE COMPARISON

Let (M, n) block error correction code denoted with \mathcal{C} be defined as a map $\phi_{\mathcal{C}} : i \rightarrow x_i$ from the set $i \in \Theta = \{1, \dots, M\}$ of possible information messages to the set of codewords $x_i \in \mathcal{A}^n$ for $i = 1, \dots, M$. Let \mathcal{A} denote the alphabet of the code symbols.

Let the received message be z if the transmitted code word is $x = \phi_{\mathcal{C}}(y)$ as a result of encoding the information message y . The dependence of received message z on x is described with a probabilistic law, examples of which will be discussed in more detail in the following sections.

The original transmitted message y is recovered through the usage of *decoder* δ which maps the received message z to a possible transmitted data message y

$$\delta(z) \rightarrow y. \quad (4)$$

In the information theory as well as communications practice, a common metric used to evaluate the performance of the block error correction code is the packet (code-word) error probability. Let the transmitted message be y . Then, the packet error probability of code \mathcal{C} is equal to

$$P_e^{\mathcal{C}}(y) = \mathbb{E} [I_{\{\delta(z) \neq y\}}] \quad (5)$$

where expectation is over realizations of the random variable z of the received code word given transmitted data message y .

A. Comparison of Error Correction Codes in AWGN Channels Based on Linear Normal Experiments

In this section we will address the transmission of the coded message over an additive white Gaussian noise channel. Namely, assume that code word $x \in \mathcal{R}^n$ is being transmitted and that the alphabet of the code symbols is \mathcal{R} . The code words are assumed to be energy bounded such that $xx' \leq E$ where parameter E is the upper bound on the energy of the code word. The received data is given as

$$z = x + \nu \quad (6)$$

TABLE I

EQUIVALENT TERMINOLOGY USED IN STATISTICAL EXPERIMENTS AND ERROR-CORRECTION CODES

Statistical Experiments	Error-Correction Codes
parameter θ	information message
decision rule	decoding algorithm
sample space	set of received code words
loss function	rate-distortion measure

where ν is the additive Gaussian noise with covariance matrix $E[\nu\nu'] = \sigma^2 I$. The signal to noise ratio is therefore upper bounded with $\frac{E}{n\sigma^2}$.

For transmission in additive white Gaussian noise channel, a block error correction code can be interpreted as a linear experiment $\mathcal{E}(A, I\sigma^2; \beta \in \mathcal{R}^M)$ such that the parameter set is $\Theta = \mathcal{R}^M$ and matrix is $A = [x_1 x_2 \dots x_M]'$. The i -th information message is represented with the parameter $\beta = e_i, i = 1, \dots, M$. Matrix A that uniquely describes such a block code will be called the *code matrix* and the adjoint block code will be denoted in short with \mathcal{A} . In the context of the comparison of experiments, decoder δ can be interpreted as the decision rule, while $I_{\{\delta(z) \neq \beta\}}$ is the loss function, for a received message z and transmitted data message β . A comparison of error-correction code and statistical experiments terminology is shown in Table I for easy reference.

Proposition 1: (Comparison of two codes based on the packet error probability) Let A and B be two code matrices that define block error correction codes \mathcal{A} and \mathcal{B} respectively. (a) If noise variance σ^2 is a known parameter and if

$$AA' - BB' \text{ is non-negative definite matrix} \quad (7)$$

then there exists a decoder of the code \mathcal{A} that will always have smaller bit error rate than any decoder of the code \mathcal{B}

$$P_e^{\mathcal{A}}(\beta) \leq P_e^{\mathcal{B}}(\beta) \quad (8)$$

for any transmitted data message $\beta = e_i, i \in 1, \dots, M$.

(b) If the above condition is not satisfied and if $\text{range}(B) \subseteq \text{range}(A)$ then for any data message β

$$P_e^{\mathcal{A}}(\beta) - P_e^{\mathcal{B}}(\beta) \leq \delta(\mathcal{A}, \mathcal{B}) = \|N(0, (B'(AA')^+ B) \vee I) - N(0, I)\| \quad (9)$$

where $P_e^{\mathcal{A}}(\beta)$ and $P_e^{\mathcal{B}}(\beta)$ are packet error rates of block codes \mathcal{A} and \mathcal{B} respectively.

Proof: The proof follows from the definition of the deficiency distance and Theorem 2. Let $\Theta_0 = \{e_i | i = 1, 2, \dots, M\}$ and the decision space be $\mathcal{Y} = \{1, 2, \dots, M\}$, while the loss function be defined as

$$\mathcal{L}(\theta, y) = I_{\{y \neq \max_i \theta_i\}}, \theta = [\theta_1, \dots, \theta_M] \in \Theta_0, y \in \mathcal{Y}. \quad (10)$$

Expected value of the above loss function corresponds to the risk function that is equal to the block error probability when averaged over random noise realizations, i.e.,

$$P_e(x) = \int P_\theta(x) \mathcal{L}(\theta, \sigma(x)) dx. \quad (11)$$

According to the Definition 2, the statement (a) of this proposition immediately follows from Theorem 2(a). Similarly, (b) directly follows from Theorem 2(b) where the Definition 1 is applied for the loss function in (10) and by noting that $\max_y L(\theta, y) = 1$. \square

Remark 1: (Influence of the Information Message Distribution) Note that the conclusions of the Proposition 1 are valid for *any* data message $\beta = e_i, i = 1, \dots, M$. Therefore, it follows that the code comparison results of Proposition 1 are valid for any a priori distribution of data messages.

Remark 2: (Loewner Order and Moment Generating Matrices) Matrix partial order $A \geq B$ whenever $AA' - BB'$ is non-negative definite is commonly referred to as Loewner order. In the experiment design literature matrix AA' is commonly referred as the moment matrix of a linear experiment $\mathcal{E}(A, \Sigma; \beta \in \mathcal{R}^n)$.

Remark 3: (Bayes risk) The conclusions of the Proposition 1 are valid for the comparison of two codes with respect to *any* loss function and not just the packet-error probability. Namely, any Bayesian risk function can be used and the conclusions of the Proposition still hold.

Remark 4: (Limitations of the code comparison based on linear experiments) The code comparison based on Loewner order is very general and strong. There are two reasons for this statement: (i) with respect to the block code, parameter set of the adjoint linear experiment is extended from a finite set to the n -dimension space \mathcal{R}^n , and (ii) the comparison is valid for any risk function and may encompass risk function that may not be of interest in the code design and performance analysis. Therefore, for some applications the packet error rate bound based on the deficiency bound may be loose.

Next, we give an example of the code comparison based on the linear experiment comparison. The codes compared have the same parameter set.

Example 1: (a) BCH(63,7) code is better than BCH(15,7) code for any loss function. (b) However, code comparison between BCH(31,7) and BCH(15,7) cannot be established based on Proposition 1. These

conclusions are easily established by calculating their respective code matrices and checking if Loewner ordering holds.

B. Comparison of Error Correction Codes in Discrete Channels: Matrix Majorization and Zonotopes

In this section we assume that the code symbol alphabet \mathcal{A} is finite with l elements. Further, we assume that channel outputs also belong to the finite alphabet of w symbols, i.e. that the channel is discrete. In light of the experiment comparison terminology, therefore w^n possible received noisy code-words are elements of the sample space \mathcal{X} , while $M = 2^k$ information messages (of length k bits) are elements of the parameter set Θ .

A finite code symbol alphabet block error correction code \mathcal{E} can be represented with a row-stochastic $2^k \times l^n$ code matrix $M_{\mathcal{E}}$. The code word $\phi_{\mathcal{E}}(i)$ corresponding to the information message i is the element of the set \mathcal{A}^n . Row i of $M_{\mathcal{E}}$ corresponds to the information message $i \in 1, \dots, 2^k$ and is equal to $e_{\phi_{\mathcal{E}}(i)}$, i.e. it is equal to the unity vector with 1 in $\phi_{\mathcal{E}}(i)$ -th place. As will be demonstrated below using Theorem 3, the ordering of information and code messages is arbitrary and our proceeding results do not depend on this ordering.

Channel \mathcal{C} is modeled as follows. Since the number of observations is considered to be finite, the channel is modeled with a row stochastic $l^n \times w^n$ matrix $C = [p_{ij}] \in \mathcal{M}_{l^n w^n}$. Probability p_{ij} is the probability of receiving i -th element of the sample space \mathcal{X} if code word $j \in \mathcal{A}^n$ is being sent. This general channel description incorporates binary symmetric channels (BSC) (as discussed in the Example 2) as well as many other channels such as bursty error channels.

Following the naming conventions introduced in Section I, coding experiment will be considered as the observation of the parameter (or message) $\theta \in \Theta$ in channel \mathcal{C} after coding the message with the block code \mathcal{E} . Therefore, the probability measure P_{θ} of such experiment on the finite sample space \mathcal{X} is given with the rows of the *transfer matrix* $M_{\mathcal{E}}C$. Using the theory of experiment comparison, performance comparison of two block codes \mathcal{E} and \mathcal{F} in the same channel \mathcal{C} can be based on their transfer matrices $M_{\mathcal{E}}C$ and $M_{\mathcal{F}}C$.

Let us consider the channel \mathcal{C} . Then, code \mathcal{E} is sufficient for code \mathcal{F} if $M_{\mathcal{E}}CM = M_{\mathcal{F}}C$, for some $w^n \times w^n$ dimensional row-stochastic matrix M [1]. This condition is equivalent to the matrix majorization [7] of matrix $M_{\mathcal{E}}C$ with respect to $M_{\mathcal{F}}C$ and is also denoted with $M_{\mathcal{E}}C \succ M_{\mathcal{F}}C$. Therefore, for any loss

function and decoder associated with decoding of the code \mathcal{F} , there exists a decoder for the code \mathcal{E} that produces less or equal risk. In addition, under the above majorization condition, code \mathcal{E} can have smaller code word error probability than the code \mathcal{F} .

Several properties of the matrix majorization are shown in [7]. This article also demonstrates that it is possible to check the matrix majorization ordering between two matrices by checking the feasibility of a linear program.

It is stated that the linear operator $T : \mathcal{M}_{nm} \rightarrow \mathcal{M}_{nm}$ preserves the matrix majorization ordering if $A \succ B \implies T(A) \succ T(B)$ for some $A, B \in \mathcal{M}_{nm}$.

Theorem 3: (Preservation of the Matrix Majorization Ordering [8]) A linear operator $T : M_{nm} \rightarrow M_{nm}$ preserves the matrix majorization ordering if and only if $T(X) = LXP$, where $L \in M_{nn}$ is an invertible matrix and $P \in M_{mm}$ is a permutation matrix.

An interpretation of the Theorem 3 in the context of the error correction code comparison is as follows. Since, matrix majorization is preserved if matrices are multiplied from the right by any permutation matrix P , it is obvious that code comparison ordering is preserved for any permutation of the received code words in the transmission channel. Also, let us first consider the case when invertible matrix L in the above theorem is also a permutation matrix. Then, it follows directly from Theorem 3 that code comparison ordering is preserved for any permutation of the input data messages. The general case of any invertible matrix L is not of interest as it would amount to randomization of data messages at the input of the encoder.

Due to the large dimensions of the matrices $M_{\mathcal{E}}C$ and $M_{\mathcal{F}}C$ it might not computationally be simple to check if matrix majorization ordering \succ can be established between these two transfer matrices. Therefore, to simplify the setting and to provide more insight into the matrix majorization ordering we will be considering next the special case of detecting a single bit of the information message θ consisting of k bits.

To accomplish that, we first have to introduce our following assumption:

A1: The a priori probability distribution of data messages $p(\theta)$ is known.

This assumption is warranted in practical systems. For example, source coding is usually used prior to error correction coding which renders information messages uniformly distributed.

Let us concentrate on the decoding of the r -th bit of the k -bit long information message $\theta = [b_1 b_2 \cdots b_r \cdots b_k]$.

Therefore bit b_r is the parameter of the experiment and the parameter set is $\Theta = \{0, 1\}$. All other bits in the information message are to be considered to be nuisance parameters. In the long term, the effect of the nuisance parameters can be averaged out by introducing the *equivalent code matrix* $M_{\mathcal{E}}^r$ as follows. Let $\Theta^r(0) = \{\theta = [b_1 b_2 \cdots b_r \cdots b_k] | b_r = 0\}$ be the set of information messages for which the r -th bit is equal to 0. Similarly, let $\Theta^r(1) = \{\theta = [b_1 b_2 \cdots b_r \cdots b_k] | b_r = 1\}$ be the set of information messages for which the r -th bit is equal to 1.

The equivalent $2 \times w^n$ transfer matrix $\tilde{M}_{\mathcal{E}}^r$ for r -th bit will be defined as

$$\tilde{M}_{\mathcal{E}}^r = \begin{bmatrix} \sum_{\theta \in \Theta^r(0)} p(\theta) M_{\mathcal{E}}(\theta, :) \\ \sum_{\theta \in \Theta^r(1)} p(\theta) M_{\mathcal{E}}(\theta, :) \end{bmatrix}. \quad (12)$$

The coefficients of this equivalent transfer matrix will be denoted with $\tilde{e}_{jk}^r; j = 1, 2; k = 1, \dots, w^n$.

By introducing the equivalent experiment for each of the bits of the information message θ , we can represent the block code \mathcal{E} as a sequence of k experiments with parameter set of just two elements. In the statistical literature the experiment with the parameter set of cardinality 2 is usually called a *dichotomy*. Dichotomy will be denoted with $\mathcal{D} = \{D, \theta \in \{0, 1\}\}$, where D is a 2-row stochastic transfer matrix. In the case of dichotomies, the matrix majorization has several very useful simplifying properties. However, to be able to use these properties, we first have to introduce the concepts of zonotopes.

Definition 4: (Dichotomy Zonotopes [7]) Consider a dichotomy $\mathcal{D}_A = \{A, \theta \in \{0, 1\}\}$ where $A = \{a_{ij}\}$ is a $2 \times n$ dimensional row stochastic matrix. The zonotope of the dichotomy \mathcal{D}_A is defined as

$$Z(\mathcal{D}_A) = \sum_{i \in \{1, \dots, n\}}^{\oplus} \text{conv}(0, A(:, i)) \quad (13)$$

where the addition \oplus in the previous equation is the Minkowski addition.

Recall that Minkowski addition \oplus of two sets is the set of sums of all possible combination of elements from these two sets, i.e.

$$\mathcal{A} \oplus \mathcal{B} = \{a + b | a \in \mathcal{A}, b \in \mathcal{B}\} \quad (14)$$

Dahl [7] showed that the dichotomy zonotope is a polygone that contains the origin and is symmetric with respect to the point $(\frac{1}{2}, \frac{1}{2})$. The upper boundary of the dichotomy zonotope can be calculated as

$$\beta_{\mathcal{A}}(x) = \max_y \{y | (x, y) \in Z(\mathcal{A})\} \quad (15)$$

$$= \max \left\{ \sum_{j=1}^n a_{2j} \delta_j \mid \sum_{j=1}^n a_{1j} \delta_j \leq x, 0 \leq \delta_j \leq 1 \right\} \quad (16)$$

for $x \in [0, 1]$. Now, two dichotomies can be compared as follows:

Theorem 4: (Comparison of Dichotomies [7]) Let $\mathcal{A} = \{A, \theta \in \{0, 1\}\}$ and $\mathcal{B} = \{B, \theta \in \{0, 1\}\}$ be two dichotomies with the same parameter set. Then, the following properties are equivalent:

- (a) $\mathcal{A} \leq_S \mathcal{B}$
- (b) $A \prec B$
- (b) $Z(\mathcal{E}) \subseteq Z(\mathcal{F})$
- (c) $\beta_{\mathcal{A}}(x) \leq \beta_{\mathcal{B}}(x), x \in [0, 1]$.

Proof: Corollary 4.2 in [7]. \square

Now, using the per-bit equivalent matrix representation of the block code experiment we can state the following corollary regarding the bit-error probabilities of the decoding of a particular bit of a code word.

Corollary 1: (Per-bit Code Comparison) Let \mathcal{E} and \mathcal{F} be two block error correction codes used in channels \mathcal{C}_1 and \mathcal{C}_2 with equivalent transfer matrices $\tilde{M}_{\mathcal{E}}^r = \{\tilde{e}_{jk}^r\}$ and $\tilde{M}_{\mathcal{F}}^r = \{\tilde{f}_{jk}^r\}$, respectively. Then, the probability of decoding the r -th bit of the information message of code \mathcal{E} can always be less than that of decoding code \mathcal{F} if

$$\max_{\left\{\sum_{j=1}^n \tilde{e}_{2j}^r \delta_j \mid \sum_{j=1}^n \tilde{e}_{1j}^r \delta_j \leq x, 0 \leq \delta_j \leq 1\right\}} \geq \quad (17)$$

$$\max_{\left\{\sum_{j=1}^n \tilde{f}_{2j}^r \delta_j \mid \sum_{j=1}^n \tilde{f}_{1j}^r \delta_j \leq x, 0 \leq \delta_j \leq 1\right\}} \quad (18)$$

We next illustrate the shape and certain properties of the zonotope of a block error correction code.

Example 2: (BCH Code Comparison using Zonotopes) Consider the use of Hamming(15,7) code \mathcal{E} in a binary symmetric channel (BSC) with probability of error p . In Figure 1, we show zonotopes for $Z_1(\mathcal{E})$ and $Z_2(\mathcal{E})$ for the use of this code in BSC with probabilities of error $p_1 = 0.1$ and $p_2 = 0.2$, for the first information bit. It is obvious that the $Z_1(\mathcal{E}) \supset Z_2(\mathcal{E})$ and that code \mathcal{E} will be better performing in the channel with error probability p_1 , than in the channel with error probability p_2 .

Remark 5: The error-correction comparison based on dichotomies and bit-error rates is more specific than the comparison based on the comparison of the linear experiments. This is due to the fact that comparison based on dichotomies is using more information about the structure of the parameter set to be used to convey information. As opposed to that the parameter set in the case of linear experiments is the set \mathcal{R}^k even if the true set of information messages is finite set $\Theta = \{0, \dots, M\}$.

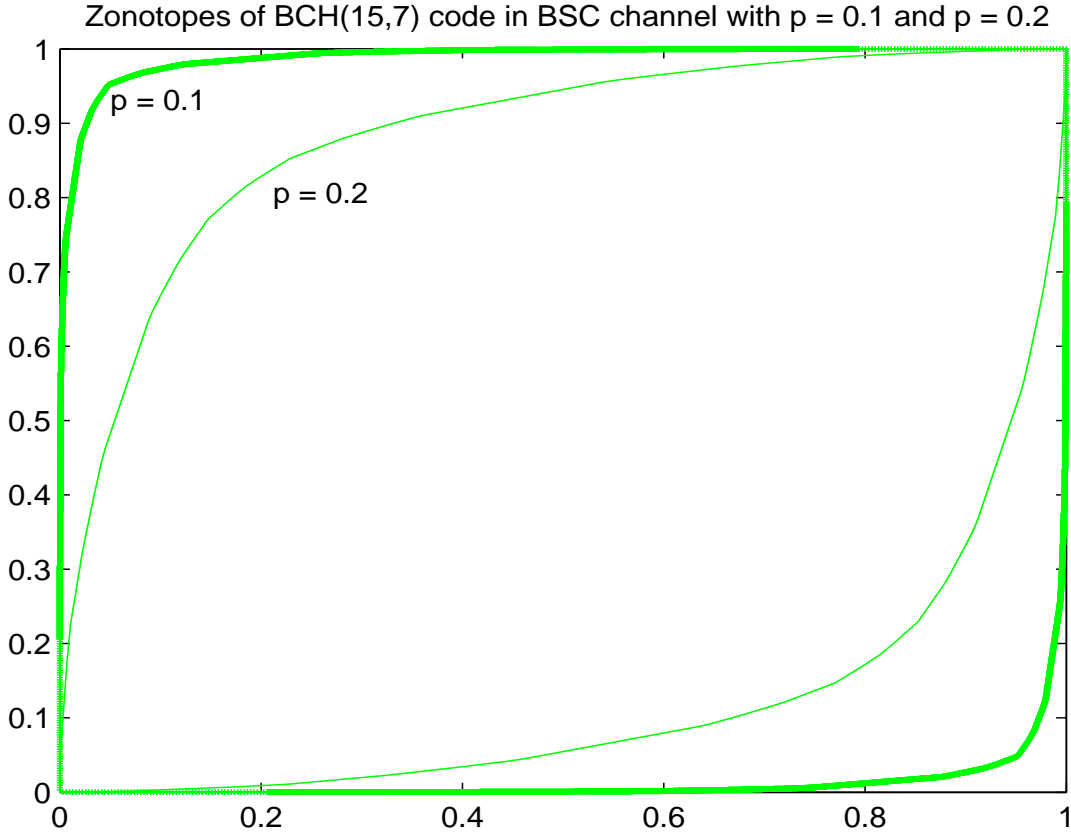


Fig. 1. Illustration of the zonotopes of the BCH(15,7) codes in the BSC channel with $p = 0.2$ and $p = 0.1$.

III. EXTENSIONS AND CONCLUSION

In a similar manner one can compare the performance of the spreading codes in CDMA systems or compare performances of two distinct Multiple-Input Multiple-Output (MIMO) channels. For example, consider a MIMO channel

$$y = Ax + n \quad (19)$$

where x is a t -dimensional column vector of the transmitted message, y is a r -dimensional column vector of the received message, A is $t \times r$ -dimensional channel matrix, and n is the additive Gaussian noise with covariance matrix I_r .

It is obvious that results of the Theorem 2 can be directly applied and the channels can be compared using the concepts of Loewner ordering and deficiency. For example, MIMO channel defined with channel matrix A is better than MIMO channel defined with channel matrix B iff $AA' - BB'$ is non-negative definite. The criterion for performance comparison can be any loss function for the estimation of the

uncoded transmitted signal x . Results with a similar flavor have been obtained in [5] for the comparison of two Rician MIMO channels but without the use of powerful theory of the comparison of experiments.

In conclusion, let us mention that the elegant statistical results on the comparison of experiments and deficiencies [3], [1], [2] have been known for several decades. However, to the best knowledge of the author, despite their appeal to the problems of information transfer these results have received limited attention in the information and communication theory before.

REFERENCES

- [1] E.Torgersen, *Comparison of Experiments*, 1st ed. Cambridge University Press, 1991.
- [2] L. Le Cam, "Sufficiency and approximate sufficiency," *Ann. Math. Statist.*, vol. 35, pp. 1419 – 1455, 1964.
- [3] D. Blackwell, "Comparison of experiments," *Proc. Second Berkeley Sympos. Math. Statist. Probab.*, pp. 93–102, 1951.
- [4] C.E.Shannon, "A note on a partial ordering for communication channels," *Information and Control*, vol. 1, pp. 390–397, 1958.
- [5] D. Hoesli, Kim Young-Han, A. Lapidoth, "Monotonicity results for coherent MIMO Rician channels," *IEEE Trans. Inform. Theory*, vol. 51, pp. 4334–4339, 2005.
- [6] D.V.Djonin and V.K.Bhargava, "On the influence of the power delay profile on the performance of diversity combining systems," *IEEE Trans. on Wireless Comm.*, vol. 3, pp. 1854–1861, 2004.
- [7] G. Dahl, "Matrix majorization," *Linear Algebra Appl.*, vol. 288, pp. 53 – 73, 1999.
- [8] A.M. Hasani, M. Radjabalipour, "On linear preservers of (right) matrix majorization," *Linear Algebra and its Applications*, vol. 423, pp. 255–261, 2007.